effect is not believed to contribute significantly to the problem.

To overcome the nucleation problem, a teflon powder was substituted for the DOP. Teflon was chosen because of its nonwetting property. The particles were found to be approximately one-half micron in diameter, a size sufficiently small to ensure accurate flow tracking. Particle sizes were estimated by looking at samples with an electron microscope. Particles this size were generated by grinding the much larger (20-100 μm) particles as supplied by the manufacturer. Teflon particles are typically 0.5 μm but agglomerate during packaging and shipping. The particles were injected into the plenum. The results obtained agreed with pneumatic data on both sides of the oblique shock to within two percent (Fig. 2). This agreement was well within the combined accuracies of the two measuring systems. Teflon powder therefore appears to be a suitable LV seeding material in applications where predried air cannot be provided.

References

[1] Yanta, W.J., "Measurements of Turbulence Transport Properties with a Laser Doppler Velocimeter," AIAA Paper 73-169, Washington, D.C., 1973.

[2] Johnson, D.A. and Rose, W.C., "Turbulence Measurements in a Transonic Boundary Layer and Free-Shear Layer Using Laser Velocimetry and Hot-Wire Anemometry Techniques," AIAA Paper 76-399, San Diego, Calif., 1976.

Supersonic Wave Drag for Nonplanar Singularity Distributions

Wilson C. Chin*
Boeing Commercial Airplane Company, Seattle, Wash.

Introduction

THE wave drag for a general distribution of sources and doublets on an arbitrarily curved surface is considered. The linear differential equation of supersonic flow is assumed, with no linearized restrictions placed on the boundary conditions. An expression is derived that extends Hayes'[1] wave drag results to arbitrary curved surface doublet/source representations as are produced by recently developed panel-type computational methods.

Wave drag comprises a significant portion of the total drag in supersonic flow, and for this reason, accurate prediction methods are desirable. Hayes'[1,2] results are well known and have been accepted in general use. He examines far-field momentum considerations and, for example, arrives at a wave drag formula for a general distribution of sources without restrictions on thickness. His results showed how, for arbitrary nonlifting bodies, von Karman's[3] formula for lineal source distributions applied locally at any azimuthal station. However, in treating the effects due to lift, certain implied near-planar assumptions were made. They arose in considering the far-field momentum flux due to a distribution of horseshoe vortices, which are restricted to lie along surfaces whose generators are aligned with the streamwise axis of the governing differential equation. Recent advances in panel method computational technology (e.g., Ehlers et al.[4]) have provided methodology for completely nonplanar representation of a configuration surface by surface source and doublet distributions. These later developments also embody composite source/doublet distributions which no longer bear the classical but limiting relationships relating local source strength to local frontal area change and local vorticity strength to local lift (as an example, a fuselage can be represented entirely by a surface distribution of doublets alone, with no sources). Thus, there is a need to extend Hayes' work to handle the types of configuration representation that are made possible with the newer panel methods.

Hayes' exact result for the source problem is easily summarized. Essentially, consider sources of density $\tilde{f}(Q)$ where Q is the source coordinate. Define an equivalent density f such that

$$f(x_i;\theta)\,\mathrm{d}x_i = \iiint_{V(x_i;\,\theta)} \tilde{f}(Q)\,\mathrm{d}V \tag{1}$$

where θ is an angle measured in a plane normal to the freestream; $V(x_i;\theta)$ is the region contained between two Mach planes $x_i = x - \beta y\cos\theta_0 - \beta z\sin\theta_0$ perpendicular to a given meridian plane $\theta=\theta_0$, and intersecting the x-axis at $x=x_i$ and $x=x_i+\mathrm{d}x_i$. Note x is aligned with the undisturbed flow at infinity, and $\beta=(M_\infty^2-1)^{1/2}$ where M_∞ is the freestream Mach number. Invariance arguments suggest local application of von Karman's drag formula, i.e.,

$$\mathrm{d}D_w/\mathrm{d}\theta = -\frac{\rho_\infty U_\infty^2}{8\pi^2}\int_0^\ell\int_0^\ell f'(x_1;\theta)f'(x_2;\theta) \log_e|x_1-x_2|\,\mathrm{d}x_1 dx_2 \tag{2}$$

where ρ_∞ is the undisturbed density, U_∞ is the speed at infinity, and ℓ is the body length. The net wave drag D_w is obtained by integration, giving

$$D_w = \int_0^{2\pi} (\mathrm{d}D_w/\mathrm{d}\theta)\,\mathrm{d}\theta \tag{3}$$

The results mentioned previously were amended by Hayes to include the influence of elementary horseshoe vortices such as were used in the classical theory to represent lateral force components (lift and side-force). A function $\tilde{h}$ is defined, such that

$$\tilde{h} = \tilde{f} - \beta(\tilde{\ell}\sin\theta + \tilde{S}\cos\theta) \tag{4}$$

where $\rho_\infty U_\infty^2\tilde{\ell}$ and $\rho_\infty U_\infty^2\tilde{S}$ are lift and side forces per unit volume. As before, define

$$h(x_i;\theta) = \iiint_{V(x_i;\,\theta)} \tilde{h}(Q)\,\mathrm{d}V \tag{5}$$

Then, Eqs. (2) and (3) hold with f replaced by h. This constitutes the wave drag theory as it presently stands. It is limited to the use of elementary horseshoe vortices for lift and side-force representation, which is overly restrictive in terms of the surface modeling used with the newer panel-type computational methods.

Analysis

We consider an arbitrarily curved surface S localized in space upon which are distributed sources and doublets. The flowfield at an observation point (x,y,z) in space depends only on the singularities upstream of the intersection defined by S and the upstream Mach cone. Let the projection of this curve on the horizontal plane be C. If $z_l = z_l(x_l,y_l)$ describes S, C is defined by the solution $x_l = C(y_l)$ to the equation

$$(x-x_l)^2 - \beta^2(y-y_l)^2 - \beta^2[z-z_l(x_l,y_l)]^2 = 0$$

Received Jan. 27, 1977.
Index categories: Aerodynamics; Computational Methods; Supersonic and Hypersonic Flow.

*Specialist Engineer, Aerodynamics Research Group.

In a symbolic way, the source potential ϕ_s can be written

$$\phi_s(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C(y_1)} \frac{\tilde{f}(x_1,y_1)}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{1/2}} G_1(x_1,y_1)\,\mathrm{d}x_1\mathrm{d}y_1 \tag{6}$$

where $\tilde{f}(x_1,y_1)$ is the areal source strength, and

$$G_1(x_1,y_1)\,\mathrm{d}x_1\mathrm{d}y_1 = (1+z_{1_{x_1}}{}^2+z_{1_{y_1}}{}^2)^{1/2}\mathrm{d}x_1\mathrm{d}y_1 = \mathrm{d}S$$

is an incremental surface area. The drag due to Eq. (6) is *exactly* given by Eqs. (1-3), and we now consider the effect of doublets. Thus, examine the potential

$$\phi_D(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C} \frac{\mu(x_1,y_1;x,y,z)G_1\,\mathrm{d}x_1\mathrm{d}y_1}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{3/2}} \tag{7}$$

where

$$\mu(x_1,y_1;x,y,z) = R(x_1,y_1)(x-x_1) + \beta^2P(x_1,y_1)(z-z_1(x_1,y_1)) + \beta^2Q(x_1,y_1)(y-y_1) \tag{8}$$

Here, $R,P,$ and Q are strengths of doublets whose axes are aligned in the directions of the $x,z,$ and y axes respectively (an arbitrary doublet axis orientation is simply a linear combination of these three components). The idea is to rewrite Eq. (7) in the form

$$\phi_D = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C} \frac{\mu G_1}{2(x_1-x)+2\beta^2(z-z_1)z_{1_{x_1}}} \frac{2(x_1-x)+2\beta^2(z-z_1)z_{1_{x_1}}\,\mathrm{d}x_1\mathrm{d}y_y}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{3/2}} \tag{9}$$

and integrate by parts, so that

$$\phi_D(x,y,z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} I\,\mathrm{d}y_1 \tag{10}$$

where

$$I = \left.\frac{\mu G_1}{[x_1-x+\beta^2(z-z_1)z_{1_{x_1}}][(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{1/2}}\right|_{C(y_1)}^{-\infty} + \int_{-\infty}^{C} \frac{\dfrac{1}{G_1}\dfrac{\partial}{\partial x_1}\dfrac{\mu G_1}{x_1-x+\beta^2(z-z_1)z_{1_{x_1}}}}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{1/2}} G_1\mathrm{d}x_1 \tag{11}$$

The first term vanishes if $\mu(-\infty, y_1;x,y,z)=0$, and if we discard the "infinite part" associated with the square root singularity, this leaves

$$\varphi_D = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{C} \frac{\dfrac{1}{G_1}\dfrac{\partial}{\partial x_1}\left[\dfrac{\mu G_1}{x_1-x+\beta^2(z-z_1)z_{1_{x_1}}}\right]}{[(x-x_1)^2-\beta^2(y-y_1)^2-\beta^2(z-z_1)^2]^{1/2}} G_1\,\mathrm{d}x_1\,\mathrm{d}y_1 \tag{12}$$

which is to be compared with Eq. (6). It follows that an "equivalent source strength" Σ may be defined as

$$\Sigma = \tilde{f} + \frac{1}{G_1}\frac{\partial}{\partial x_1}\left[\frac{\mu G_1}{x_1-x+\beta^2(z-z_1)z_{1_{x_1}}}\right] \tag{13}$$

To complete the analogy to Eq. (4), we evaluate Eq. (13) at large distances from the body. For this purpose, set $y=\tilde{R}\cos\theta$, $z=\tilde{R}\sin\theta$, and $x=x'+\beta\tilde{R}$. With x' fixed and $\tilde{R}$ tending toward infinity, the asymptotic value of Σ becomes

$$\Sigma \cong \tilde{f} + \frac{\left[\begin{array}{l}\beta^2\sin\theta\cos\theta(Q_{x_1}G_1z_{1_{x_1}}+QG_{1_{x_1}}z_{1_{x_1}}-QG_1z_{1_{x_1x_1}}) \\ -(R_{x_1}G_1+RG_{1_{x_1}})-\beta\sin\theta(P_{x_1}G_1+PG_{1_{x_1}})-\beta\cos\theta(Q_{x_1}G_1+QG_{1_{x_1}}) \\ +\beta\sin\theta(R_{x_1}G_1z_{1_{x_1}}+RG_{1_{x_1}}z_{1_{x_1}}-RG_1z_{1_{x_1x_1}})+\beta^2\sin^2\theta(P_{x_1}G_1z_{1_{x_1}}+PG_{1_{x_1}}z_{1_{x_1}}-PG_1z_{1_{x_1x_1}})\end{array}\right]}{(1-\beta\sin\theta\, z_{1_{x_1}})^2G_1(x_1,y_1)} \tag{14}$$

In other words, Hayes' algorithm is modified by replacing Eq. (4) by Eq. (14). We can easily verify that Eq. (14) reduces to Eq. (4) for planar problems. This limit refers to $z_1=0$, $G_1=1$ and $R=0$, so that

$$\Sigma_{\text{planar}} = \tilde{f} - \beta\sin\theta\, P_{x_1} - \beta\cos\theta\, Q_{x_1} \tag{15}$$

For planar wings, the term P_{x_1} is proportional to the lift density $\tilde{\ell}$, and Q_{x_1} is proportional to the side-force density $\tilde{S}$. But in general, Eq. (14) must be used to calculate the drag for configurations whose surfaces are represented by nonplanar source and/or doublet distributions.

Summary

Hayes' algorithm for wave drag computation has been refined to include nonplanar effects. The only change consists in replacing Eq. (4) by Eq. (14). Recent numerical techniques in supersonic aerodynamics have centered on the use of source/doublet panel methods, e.g., Ehlers et al.[4] The present results, in terms of doublets, could be easily incorporated into these techniques. Implementation requires only the basic computer codes for Eqs. (1-3), which are already in extensive use. The total drag of an airplane configuration is obtained by adding to D_w the vortex drag of the infinite wake.

References

[1] Hayes, W.D., *Linearized Supersonic Flow,* North American Aviation, Los Angeles, Calif., Report No. AL 222, 1947.

[2] Ashley, H. and Landahl, M., *Aerodynamics of Wings and Bodies,* Addison-Wesley, Reading, Mass., 1965.

[3]von Karman, T., "The Problems of Resistance in Compressible Fluids," *Proceedings of the Fifth Volta Congress,* R. Accad. D'Italia (Rome); 1936.

[4]Ehlers, F.E., Johnson, F.T., and Rubbert, P.E., "A High-Order Panel Method for Linearized Supersonic Flow," AIAA Paper 76-381, San Diego, Calif., 1976.

Simple Eddy Viscosity Relations for Three-Dimensional Turbulent Boundary Layers

George L. Mellor*
Princeton University, Princeton, N.J.
and
H. James Herring†
Dynalysis of Princeton, Inc., Princeton, N.J.

THE technique whereby the Clauser[1] relation for an integral eddy viscosity,

$$\nu_{to} = K\int_0^\infty (u_e - u)\mathrm{d}y \tag{1}$$

forms the basis of an outer function which is comported with an inner function to describe turbulent boundary layers has now been extensively applied in many engineering research and design studies since it was first introduced[2] (in Ref. 3, a composite function of inner and outer functions is formally described). The technique has limitations but at the same time is a simple basis for surprisingly good numerical predictions of turbulent boundary-layer behavior. Equation (1) was originally conceived by Clauser for two-dimensional boundary layers where $u = u(x,y)$ and $u_e \equiv u(x,\infty)$.

For three-dimensional boundary layers, Eq. (1) has been extended by Cebecci et al.[4] (where the reader will find other relevant references) to read

$$\nu_{to} = K\int_0^\infty [(u_e^2 + w_e^2)^{1/2} - (u^2 - w^2)^{1/2}]\mathrm{d}y \tag{2}$$

Here (u,w) is the horizontal velocity vector with components in the (x,z) directions. Although Eq. (2) does reduce to Eq. (1) for 2-D flow, it cannot be correct for 3-D flow since it is not invariant to a Galilean transformation as is the turbulent field it is supposed to represent. To see this in a simple case, consider a stationary, mean flowfield whose properties are invariant in the z-direction. Then, there is no reason why the flow cannot be described in another coordinate system translating with respect to the first such that $\tilde{u} = u$ and $\tilde{w} = w - W$ where W is the (transverse) velocity of the second coordinate system. Then if

$$\nu_{to} \equiv K\int_0^\infty [(u_e^2 + w_e^2)^{1/2} - (u^2 + w^2)^{1/2}]\mathrm{d}y$$

and

$$\tilde{\nu}_{to} \equiv K\int_0^\infty [(\tilde{u}_e^2 + \tilde{w}_e^2)^{1/2} - (\tilde{u}^2 + \tilde{w}^2)^{1/2}]\mathrm{d}y$$

we would have $\nu_{to} \neq \tilde{\nu}_{to}$; in other words, the solution would depend on W which is quite arbitrary.

Received Jan. 31, 1977; revision received March 11, 1977.

Index category: Boundary Layers and Convective Heat Transfer—Turbulent.

*Professor.

†President.

All of this suggests that, instead of Eq. (2), one should write

$$\nu_{to} = K\int_0^\infty y[(\frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y})^2]^{1/2}\mathrm{d}y \tag{3}$$

which is invariant to a Galilean transformation. Furthermore, for two-dimensional flows where $\partial u/\partial y$ is monotonic, an integration of Eq. (3) by parts will yield Eq. (1). Thus Eq. (3) is a more general interpretation of Clauser's integral, eddy viscosity relation which can, in principle, accommodate three-dimensional flows and flows where $\partial u/\partial y$ is not monotonic such as wall jets.

An interpolation formula that has come to our attention (Blackadar[5]) to provide an outer continuous function and which might be a slight improvement over the older discontinuous function is

$$\nu_t = \frac{\hat{p}}{1 + \nu_p/\nu_{to}}, \quad \nu_p \equiv \kappa^2 y^2[(\frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y})^2] \tag{4a,b}$$

ν_t is the turbulent viscosity in the expression $(-\overline{u'v'}, -\overline{w'v'}) = \nu_t\,(\partial u/\partial y, \partial w/\partial y)$ and ν_p is the Prandtl mixing length expression. Solutions obtained from Eq. (4) may be matched to the law of the wall or an empirical viscous correction function of the form, $\nu\, f(\nu_p/\nu)$ where ν is the molecular viscosity and $f \sim 0$ as $\nu_p/\nu \to \infty$, may be added to Eq. (4a). Equivalently, another viscous function may be invented to multiply Eq. (4a) and which limits to unity as $\nu_p/\nu \to \infty$.

An Alternate Approach

In recent times, when we have wished to make a boundary-layer calculation with a simple eddy viscosity approach, we have used for the outer function

$$\nu_t = q\,\ell\,S_m \tag{5}$$

where

$$q^3 = B_l\,\ell\left[\nu_t\left\{(\frac{\partial u}{\partial y})^2 + (\frac{\partial w}{\partial y})^2\right\}\right] \tag{6}$$

and for the length scale,

$$\ell = \left[\frac{(\kappa y)^4}{1 + (\kappa y/\ell_o)^4}\right]^{1/4}; \quad \ell_o \equiv \alpha\frac{\int_0^\infty |y| q\,\mathrm{d}y}{\int_0^\infty q\,\mathrm{d}y} \tag{7a,b}$$

Equation (6) is a model equation representing a balance between turbulent production and dissipation and, in situations where such a balance is expected to be true, Eq. (6) should provide a good estimate of the turbulent kinetic energy, $q^2/2$. Furthermore, Eqs. (5-7) represent a first step up the ladder to more sophisticated (and complicated) models (Mellor and Herring[6]; Mellor and Yamada[7]; Briggs, Mellor, and Yamada,[8] Hanjalic and Launder[9]).

In the above model, we have determined that $S_m = B_l^{-1/3}$, $B_l = 16.6$, and, of course, $\kappa = 0.40$. As in the case of K in Eqs. (1) or (3), α in Eq. (7a) is problem dependent; for conventional boundary layers $\alpha \simeq 0.19$. One more independent empirical constant is required compared to those in Eqs. (3) and (4) but more information is provided. Furthermore, Eqs. (5-7) may be extended rather simply to include effects of wall curvature, Coriolis acceleration, and gravity in a density stratified field (So[10], Mellor[11]). For example, for two-dimensional flow with longitudinal radius of curvature, r, it is found in Ref. 11 that

$$S_m = B_l^{-1/3}\left[1 - \frac{36A_l^2}{B_l^{2/3}}\frac{R_c}{(1-R_c)^2}\right] \tag{8}$$

where $A_l = 0.92$, $R_c \equiv 2u\{\partial(ur)/\partial y\}^{-1}$. The term in square brackets in Eq. (6) must be amended to read $[\nu_t(\partial u/\partial y-$